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Migration Modified Stochastic Logistic Growth Model: Development, Characterisation Solutions and Applications to Nigeria Population

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Abstract

This study explored and proposed the migration-modified stochastic logistic growth model (MM-SLGM), developed to capture the complex dynamics of population growth under the influence of both deterministic and stochastic forces. The model incorporates intrinsic growth rate, environmental carrying capacity, net migration effects and environmental noise, highly applicable to demographic systems in developing economies like Nigeria, where migration and uncertainty play significant roles. To approximate the solution of the underlying stochastic differential equation, we employed the Euler-Maruyama method, a well-established numerical technique that allows for time-step simulation of population trajectories. Complementing this is Kernel Density Estimation (KDE) applied to the simulated outcomes to infer the empirical probability distribution of the population over time, thus enabling robust uncertainty quantification. Both were applied to Nigeria's annual population and net migration data from 1960 – 2023 obtained from data.worldbank.org and analysed with the aid of codes written in the R environment. Results from both methods were analysed and compared. The Euler-Maruyama simulation closely followed historical population trends, particularly in the earlier decades but began to underestimate observed values in more recent years, highlighting the limitations of fixed parameter models in dynamic migration contexts. KDE offered a non-parametric view of the

distribution of population outcomes, revealing a broader probabilistic spread and emphasising the range of potential future scenarios. Together, these methods offer a complementary framework with Euler-Maruyama for pathwise analysis and forecasting and the KDE for distributional insight and risk assessment. This dual approach not only enhances model interpretability but also strengthens its application for population policy design in uncertain environments.

Keywords: Euler-Maruyama, Kernel Density Estimation, Nigeria Population, Numerical Solutions, Stochastic Logistic Model, Migration Modified, Population Dynamics.

Introduction

Over the past century, the logistic growth model has stood as a foundation pillar in understanding population dynamics, ecological interactions, and resource-constrained systems. Originally, proposed by Pierre Verhulst in 1838, the logistic growth equation modelled population growth under limiting resources, capturing a sigmoidal growth trajectory that levels off at a carrying capacity (Verhulst, 1838). However, the realities of modern populations, marked by migration, socioeconomic pressures, environmental variability and policy shifts, have demanded more robust models that transcend deterministic simplicity. This evolution has given rise to stochastic logistic growth models (SLGMs) which incorporate the unpredictable, often chaotic nature of real-world population changes by embedding stochasticity into the growth framework (Marrec, Bank, & Bertrand, 2023; Boulaasair et al., 2022; Allen, 2015, Mao, 2007).

In recent decades, the stochastic logistic growth model has undergone significant refinements, particularly through the incorporation of immigration terms and noise structures that better reflect demographic volatility in developing regions (Matis & Kiffe, 2003).

Kumar & Lal (2017) studied the effect of immigration on deterministic logistic growth and concluded that population growth is affected due to the effect of immigration on both the intrinsic growth rate and the carrying capacity. Shah (2014) explored the steady-state stochastic logistic growth model and obtained a time-dependent solution for fish growth with multiplication fluctuation. Hening, Nguyen, & Yin (2018) extended the SLGM by considering its application in spatio-temporal heterogeneous and dispersed

systems with competition. Yu (2024) considered the effect of additive noise on the SDE. Ferreira (2014) explored different models of the stochastic differential equation and solved the problem by implementing the Euler-Maruyama method.

Delgado-Vences et al., (2022) made an inference for discretised stochastic logistic differential equation application in biological growth under three different cases of continuity, incomplete data and one record for each path, using the maximum likelihood estimation for the intrinsic growth rate and diffusion bridges and expectation-maximisation algorithm to estimate the diffusion parameters. However, while these studies that were implemented for the human population did not use the stochastic conditionality those that extended the stochastic fluctuating model did not consider the human population nor incorporate the population dynamics. This study therefore explores the stochastic logistic growth models and extends them by introducing a linear additive net-migration component and also implementing numerical solutions to the SDE while applying them to the human population in Nigeria.

As a refinement of the existing developments, this study further extends the SLGM by considering how net migration refines the classical model and applies the same to the human population. This led to the proposal and development of the migration-modified stochastic logistic growth model (MM-SLGM), an advanced formulation that integrates net-migration effects and random environmental fluctuations into the classical stochastic logistic paradigm. The MM-SLGM is expressed through stochastic differential equations (SDEs) where migration is modelled as a linear term and noise can be either additive or multiplicative, depending on the context. This modification acknowledges that net migration, whether forced or voluntary, can either bolster or deplete population growth, especially in developing nations like sub-Saharan Africa and Nigeria specifically where rural–to–urban migration, cross-border movements and displacement due to conflict or economic disparity are prominent (UN DESA, 2022; Adedokun & Karzanova, 2019; Oyeniyi, 2013).

Solving the proposed MM-SLGM analytically is often intractable due to the nonlinearities and stochastic components involved. As such, numerical and empirical methods have become vital. Two prominent approaches are the Euler-Maruyama method and the Kernel Density Estimation (KDE) method. The Euler-Maruyama scheme is an extension of Euler's method adapted for stochastic differential equations and provides discretised solutions to simulate population trajectories over time and has been used in various fields (Tabar, 2019; Bayram, Partal, & Orucova Buyukoz, 2018; Kloeden & Platen, 1999). It is especially useful in modelling short-term dynamics and observing sensitivity to parameter changes such as migration rate and noise intensity.

In contrast, the KDE method offers a non-parametric approach to estimating the empirical probability density function of the population size by simulating a large number of trajectories and smoothing the results. This method has been used variously in Engineering, economics, medicine and geographical studies and is instrumental in visualising the stationary distribution and in identifying the likelihood of population levels under stochastic influences (Colbrook et al., 2020; Silverman, 2018; Weglarczyk, 2018; Iacus & Yoshida, 2018; Iacus, 2008)

In the context of Nigeria, these methods are not mathematical conveniences but are tools for demographic insight. Nigeria's population dynamics are shaped by a high fertility rate, internal displacements, urban migration and economic migration across borders tagged "Japa" in recent times (World Bank, 2023; NPC 2022). By applying the MM-SLGM to Nigeria's population data, researchers and policymakers gain a nuanced view of potential growth patterns, equilibrium distributions and the stochastic risks that may affect future planning in health, education, housing and infrastructure. The Euler-Maruyama simulations allow for short-term forecasting under different scenarios while the KDE-based solutions help characterise long-term behaviours and stationary distributions, critical for anticipating pressure points and planning interventions (Wati et al., 2024; Yu, 2024; Wang, Liu, & Chang, 2022; Sole-Mari et al., 2017).

Moreover, MM-SLGM provides a theoretical framework for integrating policy-sensitive parameters. By modifying the model to include variables such as net migration and environmental shocks, one can assess not only population size trajectories but also the influence of migration control policies and conflict resolution on population equilibrium (Ejakpovi & Siloko, 2023; Marrec, Bank, & Bertrand, 2023; Agog et al., 2020; Kumar & Lal, 2017; Bacaër, 2011). In particular, when calibrated by empirical data from Nigeria, the model facilitates simulations under various policy assumptions, yielding probabilistic forecasts that are more aligned with uncertainties inherent in real-life demographic systems.

Hence, this study proposed a migration-modified stochastic logistic growth model which will be a significant advancement in population modelling by extending the classical stochastic logistic growth model with an additive net-migration rate component. By incorporating both deterministic structure and stochastic realism, and providing flexible tools like the Euler-Maruyama method and the Kernel density Estimation method for simulation and inference, the model not only deepens our technical understanding of population systems but also enhances practical decision-making. Also, applying the model to the Nigeria population data offers timely, relevant

insights at a period when demographic transitions, urbanisations and migration pressures are reshaping the nation's future.

Methodology

Development for the Classical Stochastic Logistic Growth Model

The classical stochastic logistic growth model (SLGM) captures the evolution of a population by incorporating random fluctuations into the logistic growth model by Verhulst (1838). Its use in literature has been well-documented by different authors (Marrec, Bank, & Bertrand, 2023; Shah, 2014). Its standard formulation is given as

$$dP = rP\left(1 - \frac{P}{K}\right)dt + fPdWt \quad (1)$$

Where,

r is the intrinsic growth rate, P is the population size at time t, K is the carrying capacity, f is the stochastic coefficient i.e. the intensity of environmental (multiplicative) noise and dWt is the standard Wiener's process (random error or Brownian motion) representing environmental noise

Proposed Migration Modified Logistic Growth Model (MM-SLGM)

To account for the net migration effect, we propose a modification by adding the net migration additive component to eqn. (1), and call is the Migration-Modified Logistic Growth Model (MM-SLGM) and given as:

$$dP = rP\left(1 - \frac{P}{K}\right)dt + mPdt + fPdWt \quad (2)$$

$$dP = \left[rP\left(1 - \frac{P}{K}\right) + mP\right]dt + fPdWt$$

$$dP = \left[\left((r+m)\left(1 - \frac{P}{K}\right) + m\right)P\right]dt + fPdWt \quad (3)$$

Where,

m is the net migration rate at time t

Equation 3 is a stochastic logistic growth model with:

$rP\left(1 - \frac{P}{K}\right)$ as the intrinsic logistic growth model (Verhulst, 1838)

mP is an additive deterministic net-migration term (net migration rate per term)

$fPdWt$ is the multiplicative environmental stochasticity (via Brownian motion)

This addition enables us to study population dynamics in contexts where migration significantly alters growth patterns in developing countries like the sub-Saharan African countries in general and Nigeria in particular.

The inclusion of the net migration term in the stochastic logistic growth model introduces a new dynamic. Primarily, the net migration term acts as a constant (or

time-dependent) shift in the drift component of the stochastic differential equation (SDE). Its implications include: (i) enhanced long-term population sustainability when $m > 0$, potentially pushing the population above carrying capacity, (ii) if $m < 0$, it may counteract growth, reducing the effective equilibrium size of the population and (iii) it decouples the carrying capacity from observed upper population bound, introducing migration-dependent variability. This flexibility makes the model particularly useful in explaining African demographic transitions.

Properties of the Proposed Model

The migration-modified stochastic logistic growth model (MM- SLGM) has several properties among which are:

a) It has a nonlinear drift with a linear additive migration term.
b) Its distribution is stationary under certain conditions.
c) Its stability behaviour is strongly influenced by the sign and size of m.
d) It can apply the Fokker-Planck-based analysis for density functions.
e) It is amenable to closed-form or numerical simulations and
f) It has practical application in scenarios of fluctuating migration or economic migration policies.

Solutions of the Proposed Model

The model can be solved using the gamma type distribution function, non-gamma type solutions from different noise structures, simulated population trajectory formula (using the Euler-Maruyama method or Kernel Density Estimation – KDE method), time-dependent solutions for non-stationary distributions, quasi-stationary distributions or the piecewise or hybrid distributions methods (Pelagalli, Faccio, & Casari, 2025; Iriarte et al., 2020; Gustafsson & Sternad, 2016; Murakami, 2015). This study will explore the simulated population trajectory formula using both the Euler-Maruyama method and the Kernel Density Estimation – KDE method.

Solutions of the MM-SLGM using the Euler-Maruyama Method

Rewrite the SDE in Euler-Maruyama Form

$$dP_t = \mu(P_t, t)dt + \sigma(P_t, t)dW_t, \qquad P_0 = P(0),, t \in (0, T)$$

where

a) Drift Term $\mu(P_t, t) = rP_t\left(1 - \frac{P_t}{K}\right) + mP_t$ and the
b) Diffusion term: $\sigma(P_t, t) = fP$ and
c) $P(0)$ is the initial value (population)

Discretisation Scheme of the Euler-Maruyama Method

Let

$P_n \approx P(t_n)$, (Stochastic process)

$\Delta t = t_{n+1} - t_n$,

$\Delta W_n \sim N(0, \Delta t)$, (Brownian increment)

The Euler-Maryama update becomes

$$P_{n+1} = P_n + \left[rP_n\left(1 - \frac{P_n}{K}\right) + mP_n\right]\Delta t + fP_n\Delta W_n \quad (4)$$

For small Δt, the approximation converges in the mean square sense and this method works best when $P \geq 0$, reflecting boundary conditions or log transformations may be used to enforce positivity in practice (Ganiyu, Famuagun, & Akinremi, 2018; Bayram et al., 2018). This simulation shows the population trajectory under the combined effects of

i. The logistic growth with crowding effect $1 - \frac{P}{K}$
ii. Net migration mP and
iii. Multiplicative noise $fPdW_t$, representing environmental fluctuations

This stochastic term ensures each run of simulation can produce different trajectories, illustrating random variability in population over time.

Solutions of the MM-SLGM using the Kernel-Density Estimation Method

Since SDEs are random variables or stochastic processes, rather than solving for a single path, it ay of interest to estimate the probability distribution of the process at a given time. An effective non-parametric approach to approximate the time-dependent or stationary of an SDE solution is the Kernel Density Estimation (KDE) method (Lopez & Sanz, 2020). This method relies on simulating multiple sample paths of the SDE and estimating the PDF using kernels.

Consider eqn. (4), instead of solving for the exact distribution analytically, it is important to:

i. Simulate multiple sample paths ${P_t}^{(1)}, {P_t}^{(2)}, \ldots, {P_t}^{(n)}$ using numerical methods (such as the Maruyama)
ii. Estimate the PDF of P_t from these samples using Kernel Density Estimation

Estimating the KDE,

Given n independent samples $p_1, p_2, \ldots, p_n$, the KDE of the PDF p(x) is given as:

$$\hat{p}_h(p) = \frac{i}{nh}\sum_{i=1}^{n} K\left(\frac{p - p_i}{h}\right) \quad (5)$$

Where,
K(.) is a kernel function (usually a symmetric PDF such as the Gaussian), K > 0 is the bandwidth (smoothing parameter and $\hat{p}_h(x)$ is the estimated density (Weglarczyk, 2018).
The commonest used kernel because of its smoothness is the Gaussian Kernel given as:

$$K(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \quad (6)$$

The steps adopted are:
Firstly, simulate the SDE paths using a numerical method like the Euler-Maruyama as used in eqn. (4). Then, collect the endpoint values for a fixed time T, and simulate M trajectories $\{P_T{}^{(1)}, P_T{}^{(2)}, \dots, P_T{}^{(M)}\}$. These values represent a sample from the distribution of P_T. Lastly, apply the KDE to the sampled values using eqn. (5).
The KDE is a non-parametric method which does not assume any fixed form of distribution and can be used with any simulated data from an SDE because of its flexibility and is very effective for stationary and time-varying densities. However, it requires a large number of simulations for smooth estimation, bandwidth selection is sensitive and affects the bias-variance tradeoff and performs poorly in high dimensions (called the curse of dimensionality)

Simulation Errors

These are widely used measures in statistics, numerical modelling and forecasting to assess the accuracy of predictions and simulations (Chai & Draxler, 2014; Willmott & Matsuura, 2005; Higham, 2001). The two methods of solution were compared based on the error analysis including the absolute error (AE), relative error (RE), the means absolute error (MAE) and the root mean square error (RMSE). For a given year t_i, these errors are computed thus:

Absolute Error (AE)

The absolute error (AE) is the absolute difference between the simulated and observed population values and is given as

$$AE(t_i) = |P_{sim}(t_i) - P_{obs}(t_i)| \quad (7)$$

Where,
$P_{sim}(t_i)$ is the simulated population at the time t_i
$P_{obs}(t_i)$ is the observed population at the time t_i

Relative Error (RE)

The Relative Error (RE) gives the percentage deviation from the observed value and is expressed as:

$$RE(t_i) = \frac{|P_{sim}(t_i) - P_{obs}(t_i)|}{P_{obs}(t_i)} \quad (8)$$

Mean Absolute Error (MAE)

The Mean Absolute Error (MAE) is the average of the absolute errors over all N time points and is given as

$$MAE = \frac{1}{N}\sum_{i=1}^{N}|P_{sim}(t_i) - P_{obs}(t_i)| \quad (9)$$

Root Mean Square Error (RMSE

The root mean square error (RMSE) is the square root of the average square error and penalises large errors more heavily than MAE. It is expressed as:

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}|P_{sim}(t_i) - P_{obs}(t_i)|} \quad (10)$$

Data used in the study were obtained from the data.worldbank.org database and analysed using code written in the R environment.

Results

The result started with the application of the Euler-Maruyama method to simulate the population dynamics of Nigeria using the migration-modified stochastic logistic growth model. The simulation spans the years 1960 to 2023 using official population data and estimates a trajectory based on drift and diffusion dynamics. Kernel density estimation (KDE) was also used in simulating the MM-SLGM after estimating the probability density function (PDF) of the simulated population and comparing it against the same observed population of Nigeria.

The Euler-Maruyama method simulation used the following parameters - r = 0.025 (estimated growth rate), K = 350,000,000 (estimated carrying capacity), m = 0.000421 (mean migration rate), f = 0.07 (noise intensity) and time step = 1 year.

Table 1: The observed and simulated population values and the error statistics using the Euler-Maruyama and Kernel Density Estimation methods

Year	Observed Population	Euler-Maruyama			KDE		
		Simulated Population	Absolute Error	Percent Error (%)	Estimated Density	Absolute Error	Relative Error (%)
1960	45053782	45053782	0	0.00	9.73e-09	4.19e-09	75.52
1961	45989310	42630268	3359041	7.30	9.99e-09	4.45e-09	80.24
1962	46965292	46560347	404944	0.86	1.03e-08	4.71e-09	84.92
1963	47973575	48511387	537812	1.12	1.05e-08	4.96e-09	89.50
1964	49012016	44461413	4550602	9.28	1.08e-08	5.21e-09	93.93
1965	50086764	43649675	6437088	12.85	1.10e-08	5.44e-09	98.19
1966	51188234	49669115	1519118	2.97	1.12e-08	5.67e-09	102.22
1967	52293065	42318352	9974712	19.07	1.14e-08	5.87e-09	105.90
1968	53432783	41995636	11437146	21.40	1.16e-08	6.06e-09	109.33
1969	54631914	46658687	7973226	14.59	1.18e-08	6.24e-09	112.54
1970	55893838	44858414	11035423	19.74	1.19e-08	6.40e-09	115.47
1971	57214684	43723249	13491434	23.58	1.21e-08	6.55e-09	118.07
1972	58601748	44408305	14193442	24.22	1.22e-08	6.67e-09	120.29
1973	60082139	50032437	10049701	16.73	1.23e-08	6.77e-09	122.12
1974	61679144	48887886	12791257	20.74	1.24e-08	6.85e-09	123.49
1975	63410815	48440561	14970253	23.61	1.24e-08	6.89e-09	124.30
1976	65258009	48031532	17226476	26.40	1.24e-08	6.90e-09	124.44
1977	67234590	56504534	10730055	15.96	1.24e-08	6.86e-09	123.82
1978	69326532	66362298	2964233	4.28	1.23e-08	6.78e-09	122.35
1979	71498242	72398899	900657	1.26	1.22e-08	6.65e-09	120.00
1980	73764641	75822093	2057452	2.79	1.20e-08	6.47e-09	116.72
1981	76068103	81252520	5184417	6.82	1.18e-08	6.24e-09	112.56
1982	78378701	91325249	12946548	16.52	1.15e-08	5.97e-09	107.60
1983	80438260	87068497	6630237	8.24	1.12e-08	5.69e-09	102.56
1984	82526443	95906782	13380339	16.21	1.09e-08	5.37e-09	96.89
1985	84897973	89269896	4371923	5.15	1.05e-08	4.98e-09	89.81
1986	87235953	86984737	251215	0.29	1.01e-08	4.56e-09	82.21
1987	89591716	94178799	4587083	5.12	9.65e-09	4.10e-09	74.01
1988	92020090	86520744	5499345	5.98	9.15e-09	3.61e-09	65.06
1989	94531014	87337140	7193873	7.61	8.61e-09	3.07e-09	55.37
1990	97120925	83744046	13376878	13.77	8.04e-09	2.50e-09	45.04
1991	99720162	83873484	15846677	15.89	7.46e-09	1.91e-09	34.48
1992	102372770	69072205	33300564	32.53	6.85e-09	1.31e-09	23.64
1993	105122069	61921827	43200241	41.10	6.24e-09	6.93e-10	12.50
1994	107935714	60188406	47747307	44.24	5.62e-09	7.59e-11	1.37
1995	110819428	65367252	45452175	41.01	5.01e-09	5.32e-10	9.60
1996	113754779	65929222	47825556	42.04	4.43e-09	1.12e-09	20.15
1997	116754792	67307847	49446944	42.35	3.87e-09	1.67e-09	30.18

1998	119846254	71937860	47908393	39.97	3.35e-09	2.19e-09	39.59
1999	123047320	68967887	54079432	43.95	2.87e-09	2.68e-09	48.27
2000	126382494	71750626	54631867	43.23	2.43e-09	3.11e-09	56.10
2001	129862595	69161864	60700730	46.74	2.06e-09	3.49e-09	62.90
2002	133471989	62214120	71257868	53.39	1.75e-09	3.80e-09	68.49
2003	137202644	61816811	75385832	54.94	1.51e-09	4.03e-09	72.76
2004	141057045	65598247	75458797	53.50	1.35e-09	4.19e-09	75.67
2005	145017253	68513193	76504059	52.76	1.26e-09	4.28e-09	77.22
2006	149077334	69862819	79214514	53.14	1.25e-09	4.30e-09	77.50
2007	153267261	82989769	70277491	45.85	1.29e-09	4.25e-09	76.66
2008	157595014	87006201	70588812	44.79	1.39e-09	4.15e-09	74.93
2009	162049464	94638167	67411296	41.60	1.51e-09	4.03e-09	72.68
2010	166642886	111231160	55411725	33.25	1.64e-09	3.90e-09	70.40
2011	171379598	103099011	68280586	39.84	1.73e-09	3.81e-09	68.70
2012	176200625	97463755	78736869	44.69	1.76e-09	3.78e-09	68.23
2013	181049443	111159251	69890191	38.60	1.70e-09	3.85e-09	69.40
2014	185896915	106892555	79004359	42.50	1.54e-09	4.00e-09	72.22
2015	190671878	109015800	81656077	42.83	1.31e-09	4.23e-09	76.29
2016	195443700	119098233	76345466	39.06	1.05e-09	4.49e-09	81.07
2017	200254579	128538333	71716245	35.81	7.79e-10	4.76e-09	85.94
2018	204938755	146415628	58523126	28.56	5.44e-10	5.00e-09	90.19
2019	209485641	163935361	45550279	21.74	3.58e-10	5.19e-09	93.55
2020	213996181	178454874	35541306	16.61	2.21e-10	5.32e-09	96.02
2021	218529286	171064044	47465241	21.72	1.26e-10	5.42e-09	97.72
2022	223150896	182840475	40310420	18.06	6.66e-11	5.48e-09	98.80
2023	227882945	189122821	38760123	17.01	3.19e-11	5.51e-09	99.42

The simulated population trajectory of the Euler-Maruyama closely followed the observed population trend from 1960 to around the early 2000s. From 2010, the simulated population began to underestimate the observed values indicating that the fixed parameter values may not capture changing demographic or policy conditions over time. While the Euler-Maruyama method produced realistic stochastic trajectories, the growing divergence in later years reflects the limitations of the constant parameter models in rapidly evolving demographic contexts. However, the method effectively captures the general shape and variability of the population curve under uncertainty.

The KDE results showed an unimodal, slightly right-skewed distribution, indicating a higher likelihood of moderate to high population values within a simulated range. It also provided an estimate of the probability that each observed population value lies within the simulated distribution. These were compared to the mean density for error analysis.

Table 2: Key statistical error metrics for both methods

Error Metric	Euler–Maruyama	KDE
Mean Absolute Error (MAE)	3.38e+07	4.52e-09
Root Mean Square Error (RMSE)	4.41e+07	4.83e-09
Mean Relative Error (%)	2.54e+01	8.15e+01

Table 2 compares the performance of the two numerical methods - Euler–Maruyama and Kernel Density Estimation (KDE) in modelling and estimating the population dynamics of Nigeria from 1960 to 2023 based on a migration-modified stochastic logistic growth model. The comparison focuses on absolute and relative errors between simulated and observed data. The absolute errors of the Euler-Maryama vary widely over time especially in later years while KDE errors are smaller and more consistent. Also, the relative errors of the KDE are higher in the early years possibly due to low-density estimates at sparse data points but the Euler Maruyama has relatively better performance where population trends are smoother.

The mean absolute error (MAE) of the Euler-Maruyama is approximately 33.8million while that of the KDE is approximately 4.52e-9. In the same vein, the root mean square error (RMSE) of the Euler-Maruyama is approximately 44.1 million while that of the KDE is approximately 4.83e-9. Lastly, the mean relative error of the Euler-Maruyama is approximately 25.25.43% while that of the KDE is approximately 81.52%. Although the KDE captures the overall distribution of the population, its high relative error percentage arises because KDE estimates are densities, not values. Low-density values at observed points are more sensitive to slight deviations, making the relative error appear large even when the distribution is generally consistent.

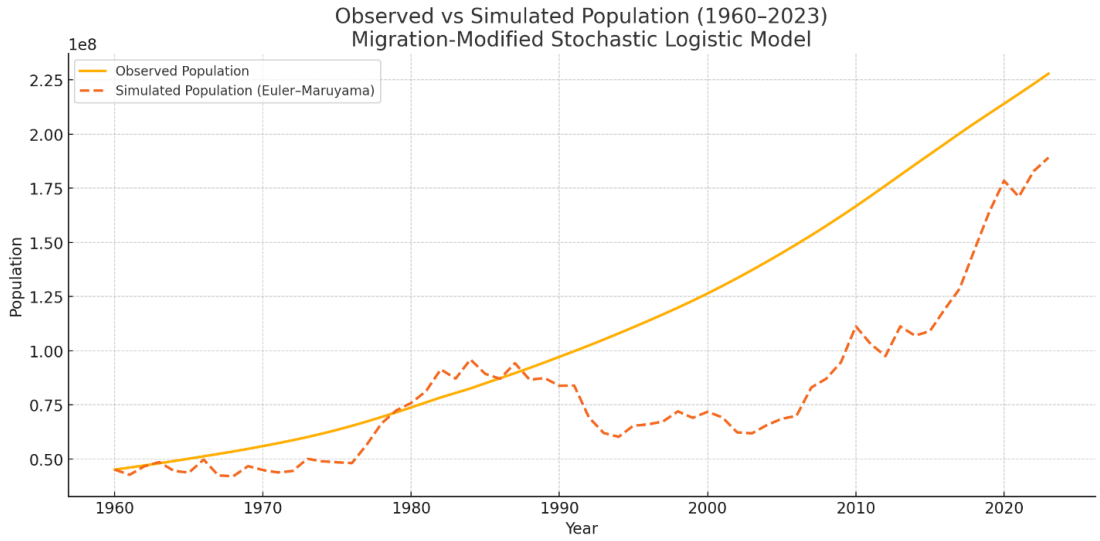


Fig. 1: Observed vs Simulated Population of the MM-SLGM over Time using the Euler-Maruyama numerical method

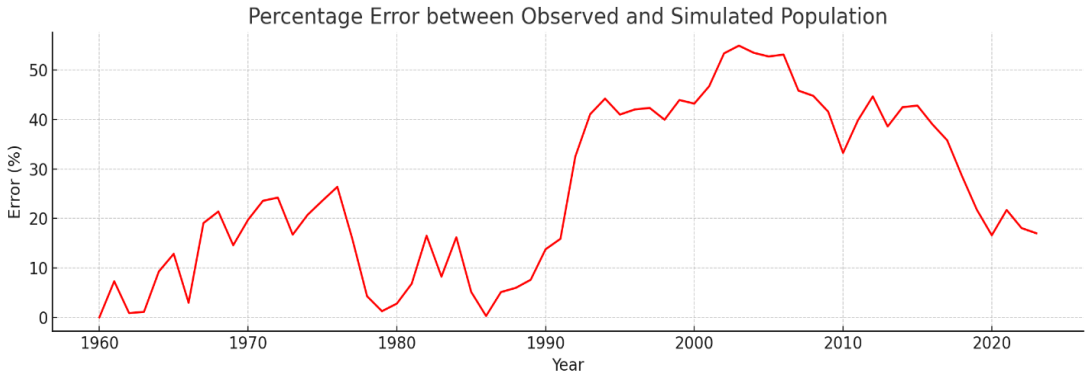


Fig. 2: Percentage Error between Observed and Simulated Population of the MM-SLGM using the Euler-Maruyama numerical method

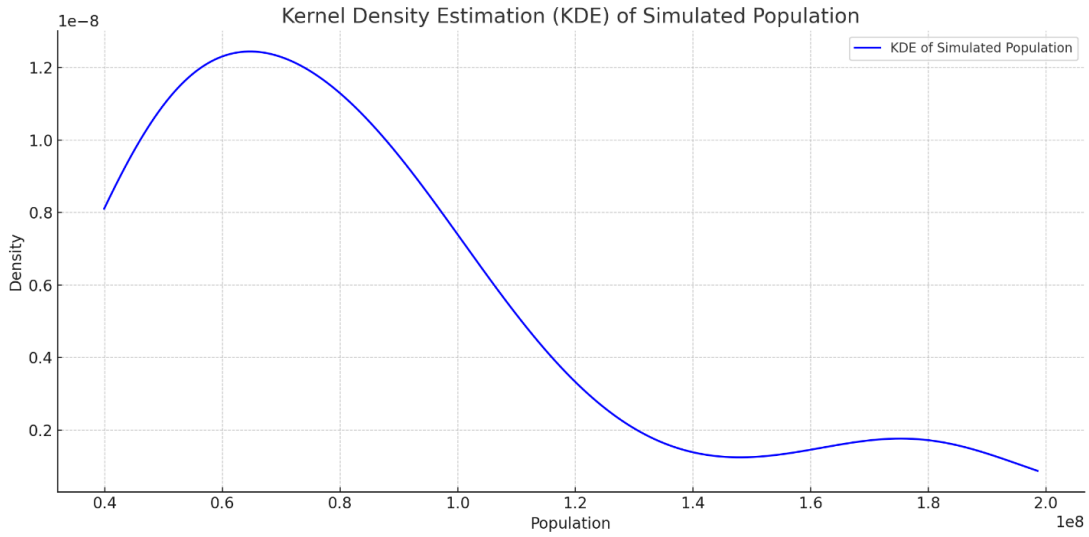


Fig. 3: Kernel Density Estimation (KDE) of the simulated population

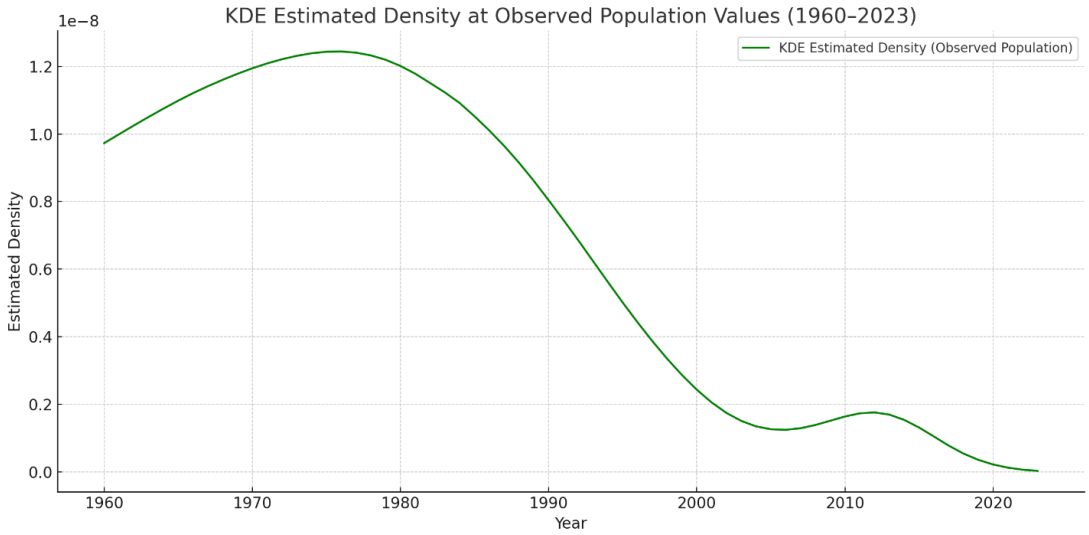


Fig. 4: KDE Estimated Densities at Observed population values (1960 – 2023)

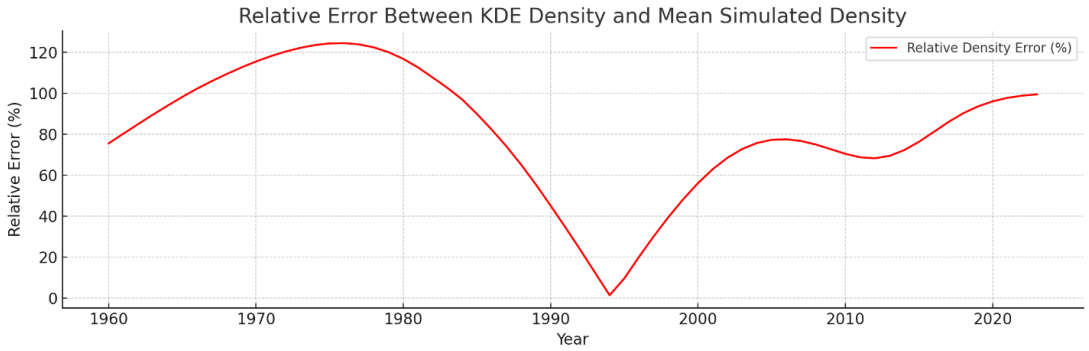


Fig. 5: Relative Error between KDE Density and Mean Simulated Density

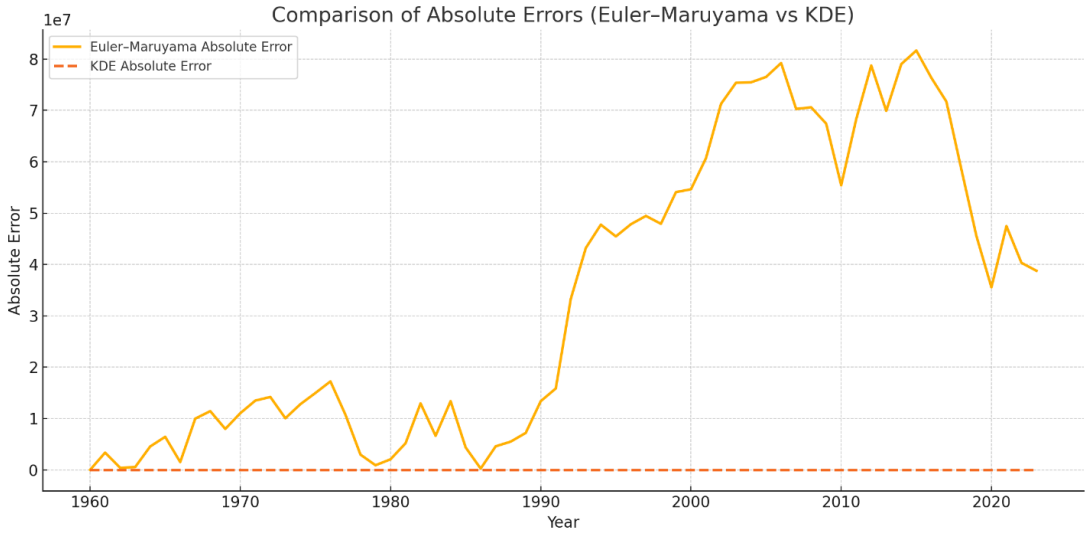


Fig. 6: Comparison of the Absolute Errors (Euler-Maruyama and Kernel Density Estimation

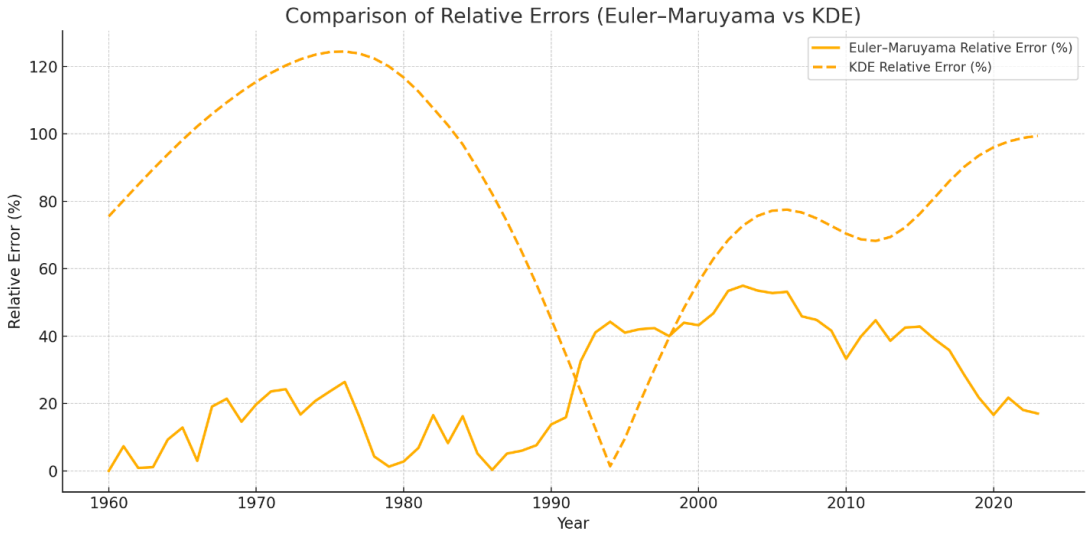


Fig. 7: Comparison of the Absolute Error (Euler-Maruyama and Kernel Density Estimation

Discussion

The Euler-Maruyama exhibited a times series trajectory and closer fit to the population values while the KDE exhibited a probability density function and fits to the overall distribution. The Euler-Maruyama has an increasing error in later years while that of the KDE has a low absolute error but high relative error at tails. The strength of the Euler-Maruyama is that it simulates dynamic growth under uncertainty while the KDE captures the population uncertainty distribution. Lastly, the Euler-Maruyama has fixed parameters which may not generalise well as its limitation while the KDE estimate depends on simulation quality and bandwidth.

From a theoretical perspective, the Euler-Maruyama simulation elucidates the sensitivity of population trajectories to net migration (m) and noise intensity (f). A higher migration term amplifies exponential components of the drift while increased environmental noise can lead to larger inert-annual fluctuations, highlighting the nonlinear propagation of demographic volatility (Abubakar, Bello, & Ismail, 2025; Hauer, Jacobs, & Kulp, 2024; Braumann, 2019; Friedman, 2010; Oksendal, 2003). This allows demographers and planners to visualise potential pathways of population change, especially under varying fertility, mortality and migration regimes.

In the stochastic logistic growth model, KDE enables researchers to move from pointwise predictions to distributional inferences (Govorov, Beconyte, & Gienko, 2025). This is especially critical when the objective is not to predict a specific population figure but to assess likelihoods, e.g., the probability that the population will exceed a critical threshold by a target year.

Conclusion

The Euler-Maruyama is very well suited for forecasting and scenario generation especially when estimating trajectories under different policy or migration scenarios while the KDE is more effective for uncertainty quantification and probabilistic analysis, especially for risk assessment, percentile estimates and decision-making under uncertainty. However, combining both approaches allows researchers to evaluate not just what will happen, but also how likely different outcomes are a more robust basis for population planning and policy modelling.

A key insight is that Euler-Maruyama and KDE are complimentary as the Euler-Maruyama generates the empirical data from a model-driven simulation while the KDE interprets the data in probabilistic terms. This synergy allows scholars to explore both dynamic (temporal) and static (distributional) properties of the population model. Also, from a theoretical standpoint, this integration bridges stochastic process modelling with empirical distribution estimation, enabling both forward simulation and backward inference (i.e., estimating likelihoods and credible intervals for population states). This duality is especially potent in Bayesian population modelling frameworks, where KDE-derived densities can serve as priors or posterior approximations.

In practical terms, these methods provide a robust foundation for evidence-based population policy. Governments can assess migration impacts under varying policy regimes (for instance, open or restricted borders). Public health planners can simulate population pressures on infrastructure under climate-induced volatility. International agencies may use KDE-based density forecasts to set confidence bounds for aid distribution or census planning. Following the outcomes, there might be a need to update the model parameters (e.g., r, m) or make the time-varying to reflect changing socioeconomic and migration conditions. Also, there might be a need to implement multi-run simulations and scenario variability. Lastly, KDE could be extended to compare empirical PDFs from real-world population snapshots for calibration.

The Euler-Maruyama and KDE methods offer more than just numerical approximations as they open a window into the complex stochastic dynamics underlying modern population change. When applied to migration-modified logistic growth models, they illuminate both the trajectories and the uncertainties that define demographic futures. Also, their combined application fosters a holistic understanding of population dynamics, integrating prediction, variability and risk assessment, a methodological imperative for 21st-century population research.

Lastly, care must be taken in consideration of the results of both simulation methods as the Euler-Maruyama methods simulated values while the KDE method simulated likelihoods and so both were not really comparable in terms accuracy and closeness of simulation. However, future studies could use methods that are comparable in values and likelihoods and the Migration modified model could be extended for other population dynamics.

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