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On the Order Product Prime Graphs for Alternating Groups of Degree Three and Four

***Dr. Muhammed Bello; **Bashir Yakubu; ***Angel Otse Ogbu; ****Fatsuma Garba Maitaya; & *****Hussaini Saidu**

*Department of Mathematics and Computer Science, Federal University of Kashere, P.M.B 0182 Gombe, Gombe State Nigeria. **A.D.Rufa'i College of Education Legal and General Studies, P.M.B 004 Misau, Bauchi State Nigeria. ***Federal College of Education (Technical) Bichi, P.M.B 3473 Kano State Nigeria. ****Federal College of Horticulture, P.M.B 108 Dadin Kowa, Gombe State Nigeria. *****Government Comprehensive Day Secondary School, Shehu Abubakar, Gombe Nigeria.

Corresponding Author: mbello@fukashere.edu.ng
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Abstract

Graph-based approaches have emerged as a key area of research in group theory, with a growing focus on leveraging graph properties to investigate algebraic structures. Research on the graphical structure of finite groups has led to the development of various definitions of group graphs over time. This study explored the relationship between alternating groups and their respective order product prime graphs. The research constructed the order product prime graphs for alternating groups of degree three and four. Regularity, completeness, connectivity, girth and diameter of each graph constructed were also checked. The study utilized the 'Group Algorithm Programming Software' to generate the elements of the alternating groups investigated, facilitating the construction of their order product prime graphs. The findings revealed that

the order product prime graphs of alternating groups of degrees three and four are connected. The study discovered that the graph of degree three has the distinctive properties of completeness, regularity, and planarity, a trait that sets it apart from the graphs of degree four. The graph of degree three has a girth of 3 and a diameter of 1, indicating a highly connected and compact structure. Notably, the graphs of degree four, exhibits a girths of 3 and diameters of 2, indicating a stable and well-connected structure.

Keywords: Group, Graph, Alternating Group, Degree Three And Four, Key Graphs' Properties

Introduction

The interplay of algebra and graph theory has captivated researchers, who have developed various methods to investigate groups through a geometric lens. This has yielded a plethora of definitions for group graphs, providing valuable tools for understanding the graphical structure of finite groups. Cayley (1878) initiated the study of groups in graph theory, paving the way for further research and the establishment of a new field dedicated to examining the properties of algebraic structure. Subsequently, the study drew the interest of researchers. Sattanathan and Kala (2009) proposed a definition for the order prime graph of a group, which allowed them to explore its properties and establish bounds on the number of edges, specifically lower and upper bounds, thereby shedding light on the graph's characteristics. Additionally, they successfully identified and elucidated the properties of specific categories of order prime graphs, further expanding the understanding of their characteristics and behavior. Bello *et al* (2020) defined a new graph associated with a finite group, called the "Order Product Prime Graph", where the group elements serve as vertices, and adjacency is determined by whether the product of their orders is a prime power, offering a novel way to explore the group's properties. Using the Order Product Prime Graph definition, Bello *et al* (2020) successfully classified finite dihedral and cyclic groups into distinct categories based on their graph properties, including connectedness, completeness, regularity, and planarity, providing valuable insights into the groups' underlying structure. This paper investigates the connection between alternating groups of small order and

their corresponding graphs, with the ultimate goal of constructing the order product prime graphs associated with these groups, thereby uncovering the graphical properties and structure of these specific groups. We introduced the concept of the order product prime graph of a group G on alternating group, which is a graph whose vertices correspond to the group elements, and where two vertices x and y are connected by an edge if and only if the product of their orders is a prime power. This definition enables us to construct the order product prime graphs of alternating groups with degree 3 and 4, providing a visual representation of the group's structural properties.

THE CONCEPTS

Definition 1(Sagan, 2013):

The symmetric group S_n is the group of all permutations of the set $\{1, 2, \ldots, n\}$. It has order $n!$

Example1 (Symmetric Group S_3)

$S_{3=\{e,(23),(12),(123),(132),(13)\}}$

Definition 2 (Miller,2012.):

The set $A_n = \{\propto \in S_n \mid \propto \text{ is even}\}$ forms a subgroup of S_n, called the alternating group on n elements.

Example 2 (Alternating Group A_3)

$A_3 = \{e, (123), (132)\}$

Theorem 1 (Wilson *et al*,2009):

Let A_n denote the set of all even permutations in S_n. Then, A_n is a normal subgroup of

S_n, with $|A_n| = |\frac{S_n}{2}| = \frac{n!}{2}$. We call A_n the alternating group.

Definition 3 (Bello *et al*, 2020):

Suppose G is a finite group, the order product prime graph of G, denoted by $\Gamma^{opp}(G)$, is a graph whose vertices are the elements of G and two vertices x and y are adjacent if and only if $|x|\ |y| = P^{\alpha}$, $\alpha \in \mathbb{N}$, where p is a prime.

Theorem 2 (Mackiw, 1995):

Every permutation can be written as a composition of transpositions. Thus, S_n is generated by transpositions.

Theorem 3 (Pommerening, 2008):

Every permutation in S_n may be written as a cycle or as a product of disjoint cycles.

Theorem 4 (Bertram & Wei, 1980):

Every permutation in S_n is a product of 2-cycles.

Definition 4 (Zhang & Chartrand, 2006):

A graph $G = (V, E)$ is a mathematical structure consisting of two sets: V, whose elements are called vertices, and E, whose elements are called edges. Each edge has a set of one or two vertices associated with it, which are called its endpoints.

Definition 5 (Van Der Hofstad, 2013):

The degree of a vertex v in a graph G, denoted by $deg(v)$, is the number of edges incident with v. In calculating $deg(v)$, a loop at v contributes 2 to the degree, not 1. A vertex with $deg(v) = 0$ is considered an isolated vertex, and a vertex with $deg(v) = 1$ is considered an end-vertex.

Definition 6 (Zhang *et al*, 2006):

A complete graph is a simple graph in which every pair of vertices is connected by an edge. A complete graph with n vertices is denoted as K_n.

Definition 7 (Zhang *et al*, 2006):

A regular graph is a graph in which all vertices have the same degree. If each vertex has a degree of r, the graph is considered regular of degree r or *r-regular*.

Definition 8 (Gutwenger *et al*, 2005):

A planar graph is a graph that can be drawn in a plane without any edge crossings, so that no two edges intersect each other except at their endpoints.

Definition 9 (Barrett *et al*, 2009):

A subgraph is a subset of vertices and edges from a larger graph, formed by deleting vertices and edges. In other words, a subgraph is a graph obtained by selecting a subset of vertices and edges from the original graph.

Definition 10 (Barrett *et al,* 2009):

A subdivision is a graph obtained by subdividing edges, which involves removing an edge and replacing it with a 2-edge path via an intermediate vertex. In other words, an edge is split into two edges by introducing a new vertex at an intermediate point.

Definition 11 (Santhakumaran & John, 2007):

In a graph G, two vertices v and w are considered adjacent if they are connected by an edge vw, and the vertices v and w are then said to be incident with this edge. Similarly, two distinct edges e and f are considered adjacent if they share a common vertex.

Definition 12 (Dehghan & Banihashemi, 2020):

The girth of a graph is the length of the shortest cycle contained in it. If a graph contains no cycles, its girth is defined to be ∞. In other words, it is the smallest number of edges that form a closed loop.

Definition 13 (Patrignani, 2013):
A path is a sequence of distinct vertices $v_1, v_2, ..., v_k$, with $k \geq 2$, together with the edges $(v_1, v_2), ..., (v_{k-1}, v_k)$. The length of the path is the number of its edges.
Definition 14 (Gorenstein, 2007):
A finite group is one with only a finite number of elements. The order of a finite group G, written $|G|$, is the number of elements in G.
Definition 15 (Chen *et al*, 2003):
A graph G is called connected if any two vertices are linked by a path.
Definition 16 (Bello *et al*, 2020):
A diameter of a graph Γ is the maximum distance between the pair of vertices of Γ.
Definition 17 (Bello *et al*, 2020):
A clique is a subset U of vertices of Γ such that the induced subgraph of U is a complete graph. The size of the maximum such clique is referred to as the clique number of Γ and denoted by $\omega(\Gamma)$. **Definition 18 (Zakariyya, 2015):**
The order of an element g, denoted by $|g|$, in a group G is the smallest positive integer n such that $g^n = e$, where e is the identity of the group. The element g has infinite order if such an integer n does not exist.

LITERATURE REVIEW

This section provides an overview of relevant concepts, including prime graphs, and order product prime graphs, all of which are crucial to understanding the core focus of this research: the order product prime graphs of alternating groups.
Research on prime graphs of groups has been undertaken by various scholars, including Akbari and Ashrafi (2015), who investigated the inclusion finite graph of certain finite groups. Their study leveraged the following theorem to derive their findings and contribute to the understanding of prime graphs in group theory.

Theorem 4 (Akbari & Ashrafi, 2015):
Let q be a power of a prime number. The proper power graphs of the following groups are disconnected:

- Sporadic groups
- Ree groups $^2F_4(q)$ and $^2G_2(q)$
- Chevalley groups $A_1(q), A_2(q), B_2(q), C_3(q),$ and $F_4(q)$
- Projective unitary group $U_3(q)$
- Projective symplectic group $S_4(q)$

Research on prime graphs of finite simple groups has been conducted by Burness and Covato (2015), who identified pairs *(G, H)* with *G* as a simple group and *H* as a proper subgroup, satisfying $\Gamma(G) = \Gamma(H)$. Staroletov (2017) defined the prime graph $\Gamma(G)$ of a finite group *G* as a graph with vertices representing prime divisors of the order of G and adjacent vertices indicating the existence of an element of order *rs*. This definition enabled the proof that if a finite group *G* has a graph equal to the alternating group's graph, then *G* has a unique nonabelian composition factor isomorphic to the alternating group of order t $(p \leq t \leq p + 3)$. Building on this work, Gorshkov and Staroletov (2019) showed that if a finite group *G* has a graph equal to the alternating or symmetric group's graph of degree n $(n \geq 19)$, then *G* has a normal subgroup with at most one prime greater than half the group's order, satisfying $A_n \leq G/K \leq S_n$.

Furthermore, Akman (2006) extended the research by computing the divisions of the alternating group and identifying the key characteristics of groups revealed by the invariant. The study yielded valuable results, including a conjecture that the invariant is a sufficient condition to distinguish finite groups, and led to the formulation of the following theorem:

Theorem 5 (Akman, 2006)

The invariant *D(G)* of any finite group *G* determines:

i. the order of G
ii. the number, orders, and relative positions of subgroups of *G* (the subgroup graph *S(G)),* therefore the positions of the intersection and join of subgroups.
iii. the positions of normal subgroups of *G*, hence whether *G* is simple or not
iv. the division graph *D(H)* of any subgroup *H of G*
v. the division graph *D(G/H)* for any normal subgroup H of G
vi. the number of divisions of *G*
vii. the positions of cyclic subgroups of *G*, possibly *G* itself
viii. the number of elements in a division δ *of G,* and the positions of the cyclic groups generated by these elements
ix. which cyclic subgroups of *G* are conjugates of each other
x. the minimum number of generators for *G*
xi. the position of the normalizer $N_G(H)$ of any subgroup *H of G*
xii. whether *G* is abelian or not, and if so, the correct composition of factors
xiii. the positions of abelian subgroups of *G*

xiv. the position of the center *Z(G)* of *G*, as well as the centralizer of any subgroup
xv. the positions of direct factors of *G*, if more than one
xvi. whether *G* is a semidirect product or not
xvii. the division graphs of a composition series of *G* and those of its unique simple factors
xviii. the position of the commutator subgroup *G′ = [G,G] of G*
xix. whether G is nilpotent/solvable

Additionally, according to Doostabadi *et al,* (2015), the power graph *P(G)* of a finite group *G* is always connected, since every vertex is adjacent to the identity element. However, when the identity element is removed from *P(G)* to form the proper power graph *P*(G),* the resulting graph is disconnected. In other words, the removal of the identity element disrupts the connectivity of the power graph, leading to a disconnected proper power graph.

Research in the literature has explored the order product prime graphs of various groups, including dihedral, cyclic, quaternion, and quasi-dihedral groups. Bello (2021) made significant contributions to this field, by introducing the graph and demonstrating that the order product prime graph is connected, complete, and regular in the following cases:

- Quasi-dihedral groups
- Cyclic groups
- Generalized quaternion groups Q_{4n}

Additionally, the graph is connected on dihedral groups of prime power degree, but only complete and regular on dihedral groups with even prime power degree.

The study also revealed that the commuting order product prime graph is connected, complete, and regular on:

- Cyclic groups of prime power order
- Quasi-dihedral groups
- Dihedral groups of prime power degree
- Generalized quaternion groups (Q_{4n} with even prime power n)

Furthermore, the order prime permutability graph of subgroups is connected, complete, and regular on:

- Cyclic groups of prime power order

- Quasi-dihedral groups
- Dihedral groups *(D_n)*
- Generalized quaternion groups *(Q_{4n}* with even prime power or prime *m*)

Other findings include the connectedness of the order prime permutability graph of cyclic subgroups on:

- Cyclic groups of prime power order
- Dihedral groups of prime degree
- Generalized quaternion groups (Q_{4n})

The study also explored the graph invariants and perfect codes of these graphs using general presentations of graphs on finite groups.

Bello *et al*, (2021) further utilized the order product prime graph to investigate graph coloring, leading to the introduction of a new graph concept: the "commuting order product prime graph of finite groups". This graph is defined as follows: its vertices represent the elements of a group *G*, and two vertices are adjacent if and only if they commute and their order product is a prime power. This extension of the order product prime graph of finite groups enabled a deeper exploration of its properties.

The commuting order product prime graph was found to exhibit various characteristics, including connectedness, completeness, regularity, and planarity, depending on the underlying group structure. These findings were formally presented and proven through a series of theorems, providing a rigorous foundation for this new area of research.

Theorem 6 (Bello *et al*, 2021)

Let *G* be a cyclic group, Z_n. Then:

$\Gamma COPP(G) \cong \Gamma OPP(G)$

The general presentation of the commuting order product prime graph on dihedral groups is given in the following.

Theorem 7 (Bello *et al*,2021)

Let *G* be a dihedral group, D_n, $n = ps, s \in N$, for some prime *p*, then

$$\Gamma^{copp}(G)=\begin{cases} 2k_1 + \left(k_{n-2} \cup \frac{n}{2} k_2\right), if\ p = 2 \\ k_1 + (k_{n-1} \cup \bar{k}_n), if\ p \neq 2 \end{cases}$$

Bello *et al* (2021) investigated the algebraic properties of groups, presenting a general framework for the graph on dihedral and cyclic groups. The study classified finite dihedral and cyclic groups based on their order product prime graphs, characterizing them as connected, complete, regular, or planar. The researchers also determined various graph invariants, including diameter, girth, independent number, and clique number. Additionally, they utilized vertex-cut to examine the nilpotency of dihedral groups. The findings revealed that:

- The graph on dihedral groups is regular and complete only for even prime power degrees, and connected for all prime power degrees.
- The graph on cyclic groups is regular, complete, and connected for prime power orders.
- The order product prime graph is planar only for dihedral groups of degree three and cyclic groups of order less than five.

These results are summarized in the following:

Theorem 8 (Bello *et al*, 2021)

$$\Gamma^{opp}(G) = \begin{cases} k_{2n} & if\ p = 2 \\ k_1 + (k_{n-1} \cup k_n), & if\ p \neq 2 \end{cases}$$

Zulkefli *et al* (2022) built upon previous research by investigating the order product prime graph of finite groups, specifically focusing on non-abelian metabelian groups of order 24 or less. Their work led to the successful construction of the order product prime graph for all non-abelian metabelian groups of order 24.

RESULT AND DISCUSSION

This section presents the findings on the Order Product Prime Graphs ($\Gamma^{OPP}(A_{3 \leq n \leq 4})$) of Alternating Groups from degree 3 to 4.

RESULTS

Order Product Prime Graph for Alternating Group of Degree Three

Consider the alternating group (A_3) of degree 3. Then, the corresponding Order Product Prime Graph is $\Gamma^{opp}(A_3) = k_3 = k_p$The graph is connected and in fact, complete, meaning every vertex is linked to every other vertex, forming a single component. With only three vertices, each vertex is adjacent to the other two, creating a triangular structure. This ensures there are no disconnected components, making the graph connected and complete. As a complete graph with three vertices, it can be represented as a triangle in a plane without edge crossings, making it planar. The planar embedding of $\Gamma^{opp}(A_3)$ is a triangle, where each vertex is

connected to the other two vertices, forming a single component. This embedding has no edge crossings, making the graph planar. Additionally, each vertex has a degree of 2, since every vertex connects to the other two, making the graph 2-regular. The graph's diameter is 1, as every vertex is directly connected to every other vertex, resulting in a shortest path of 1 edge between any two vertices. The girth of the graph is 3, as it forms a triangle, making the shortest cycle of length 3. The graph is presented in figure 1, below.

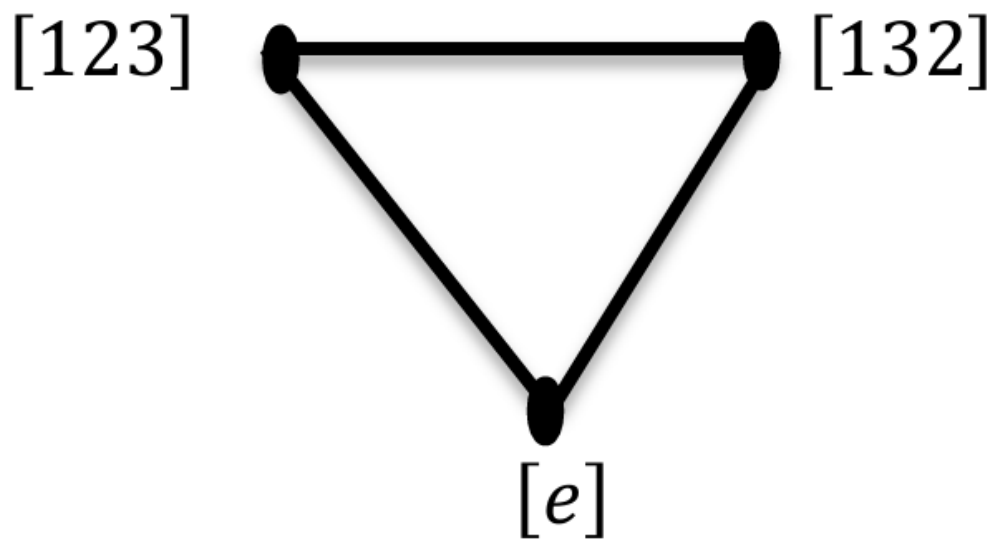


Figure 1: Order Product Prime Graph for A_3 ($\Gamma^{opp}(A_3)$)

Order Product Prime Graph for Alternating Group of Degree Four

Let A_4 be an alternating group of order 4. Then, the order product prime graph $\Gamma^{OPP}(A_4)=k_i + (k_8 + k_3) = K_1 + (k_{P_1^{r_1+1} \cup} K_{P_2})$. Where $p_1 and p_2$ are the primes respectively dividing the order of A_4 ($|A_4|$). The graph is connected, consisting of two separate components connected by identity element. The first component contains eight elements, all of which are adjacent to each other, with a distance of 1 between any two elements. The second component contains three elements. This graph is not complete, as there are vertices that are not connected to each other. The largest subgraph of $\Gamma^{opp}(A_4)$ is the first component, which contains eight elements and is a scomplete subgraph (K_8). This subgraph is a clique, meaning every vertex is connected to every other vertex, and has a diameter of 1. The second

component, containing three elements, is also a complete subgraph *(K_3),* but it is smaller than the first component. Additionally, it is not regular, as the vertices have varying degrees. The graph's diameter is 2, as the shortest path between vertices in different components is 2 edges, while within each component the shortest path is at most 1 edge. The graph's girth is 3, due to the presence of triangles (cycles of length 3) within its components. Furthermore, the graph is non-planar, as it cannot be drawn in a plane without edge crossing and has two connected components with cycles of length 3

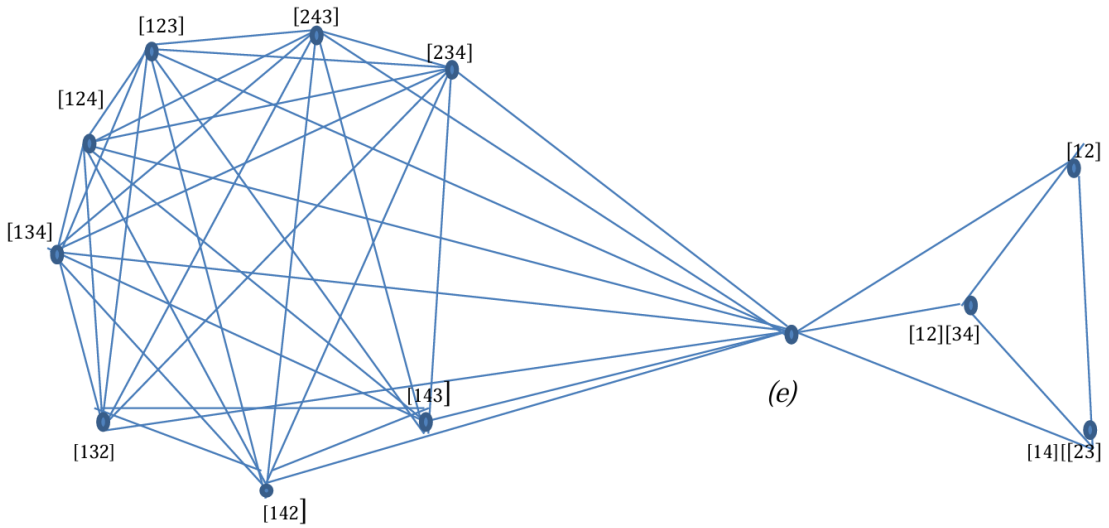


Figure 2: Order Product Prime Graph for A_4 ($\Gamma^{OPP}(A_4)$)

DISCUSSION

This study examines the order product prime graphs of alternating groups of degree 3 and 4. We analyze their connectivity, regularity, completeness, planarity, girth, and diameter. The key findings are highlighted below.

i. The order product prime graph for the alternating group of degree 3 is connected, complete, planar, and 2-regular. It has the girth of 3 and diameter of 1, aligns with Bello *et al* (2020) findings for dihedral and cyclic groups. Specifically:

- For dihedral groups:
- Regular and complete graphs were obtained for even prime powers.
- Connected graphs were obtained for all prime powers.
- For cyclic groups:
- Regular, complete, and connected graphs were obtained for all prime powers.

- Planarity was achieved when the degree of the dihedral group was 3. - Planarity was also achieved for cyclic groups with degrees less than 5.

ii. The degree 4 graph is connected with two separate components (8 and 3 vertices).

However, it is not regular, complete, or planar, with a girth of 3 and diameter of 2. Contrary to previous studies on dihedral and cyclic groups, our research reveals that applying the order product prime graph on alternating group of degree 4 yields incomplete graphs, diverging from established evidence in the existing literature.

CONCLUSION

In conclusion, this study successfully elucidated the interrelationship between group theory and graph theory by computing and analyzing the order product prime graphs of alternating groups of degree three and four, yielding novel insights into their structural complexity and its dependence on group size.

RECOMMENDATION

This paper establishes a foundation for future research into the characteristics and uses of order product prime graphs. The results can be extended to investigating higher-order graphs associated with alternating groups of greater degree.

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